LA-7291-MS

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UC-34

Issued: June 1978

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UNITED STATES
DEPARTMENT OF ENERGY
CONTRACT W-7405-ENG. 36

Printed in the United States of America. Available from
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BREAKUP OF AN ACCELERATED SHELL OWING TO RAYLEIGH-TAYLOR INSTABILITY

by

B. R. Suydam

ABSTRACT

We examine a simplified model for the Rayleigh-Taylor instability of an accelerated shell and find the most dangerous wavelength to be about that of the shell thickness. The shell material is assumed to be an inviscid, incompressible fluid. Effects of finite compressibility and of surface tension are found to be negligible, but the effects of viscosity are shown to be very large. The need for better knowledge of viscosity at high pressure is pointed out.

1. Introduction

In discussing the Rayleigh-Taylor instability of accelerated shells, it is conventional first to write down the growth rate

$$\nu = \sqrt{2\pi g/\lambda} \quad , \tag{1.1}$$

where g is the acceleration, λ is the wavelength of the perturbation, and we have assumed the density of the material accelerating the shell to be negligible compared with that of the shell material. Equation (1.1) is derived under the assumption that the shell is a perfect inviscid incompressible liquid with no surface tension. Thus as $\lambda \to o$ the growth rate $\nu \to \infty$. In spite of this obvious pathology, it is common to employ Eq. (1.1) together with the assertion: "Really, the most dangerous wavelength is that equal to the shell thickness Δ; thus in Eq. (1.1) we should set

$$\lambda \approx \Delta \quad . \tag{1.2}$$

With this value of λ we can evaluate Eq. (1.1) for breakthrough time." It is our object to discover whether a physical basis can be found for such lore.

One could argue that Eq. (1.1) is derived for a semi-infinite medium, i.e. a shell for which $\Delta >> \lambda$. In Appendix A we present the conventional Rayleigh-Taylor analysis done for a dense shell between two tenuous semi-infinite layers. The result is that, when the density of the tenuous media can be neglected, the growth rate and the mode structure are both totally independent of the thickness of the dense shell being accelerated. Thus finite shell thickness cannot be envoked to alter Eq. (1.1). Rather, we shall see from a simple phenomenological model that the nonlinear phase is responsible for singling out modes described by Eq. (1.2) as being the worst.

2. Simplified Rayleigh-Taylor Breakthrough Model

Rayleigh-Taylor instability has been described in terms of three phases:

(1) The early phase of small amplitude perturbations that grow exponentially in time as $\exp[\nu t]$. For an inviscid incompressible medium ν is given by Eq. (1.1).

(2) An intermediate or transition period, followed by:

(3) The asymptolic "bubble and spike" period. In this phase the spike grows with constant acceleration equal to g and the bubble rises at constant velocity proportional to $\sqrt{g\lambda}$.

We shall simplify first by eliminating phase (2) above. Thus our disturbance grows during phase (1) as

$$\xi = \xi_0 \, e^{\nu t} \quad , \quad \nu = \sqrt{2\pi g/\lambda} \quad , \tag{2.1}$$

where ξ is the displacement from equilibrium. According to Appendix A, this expression holds for a shell of arbitrary thickness. At time t_1 the acceleration and velocity

$$\begin{cases} \ddot{\xi}_1 = \nu^2 \, \xi_0 e^{\nu t_1} = (2\pi g/\lambda)\xi_0 \, e^{\nu t_1} \quad , \\ \dot{\xi}_1 = \nu \xi_0 \, e^{\nu t_1} = \sqrt{2\pi g/\lambda} \;\; \xi_0 \, e^{\nu t_1} \quad , \end{cases} \tag{2.2}$$

are attained.

As we are eliminating the transition phase, we are to identify the spike acceleration at t_1 with its asymptolic value g; thus the first of Eqs. (2.2) gives

$$\xi_o\, e^{\nu t_1} = \lambda/(2\pi) \quad , \quad t_1 = \sqrt{\lambda/(2\pi g)}\, \log\left[\frac{\lambda}{2\pi\xi_o}\right] . \tag{2.3}$$

Similarly we are to identify the velocity at t_1 with the bubble rise velocity. This gives

$$\dot{\xi}_1 \equiv v = \sqrt{g\lambda/2\pi} \doteq 0.40\sqrt{g\lambda} . \tag{2.4}$$

All theories and observations of bubble rise agree on a law of the form $v \propto \sqrt{g\lambda}$. The predicted values of b are rather uncertain; observed values range roughly from 0.30 to 0.35. Our crude model agrees with this reasonably well. It will, if anything, be slightly pessimistic, but not at all badly so.

Now the computation is straightforward. Let Δ represent the shell thickness. During stage (1), $0<t<t_1$, the "bubble" penetrates a distance $\lambda/2\pi$, by Eq. (2.3). Thus for the second stage there remains $\Delta-\lambda/2\pi$ to penetrate, and this at a velocity v given by Eq. (2.4). Thus the duration of the second phase, t_2, is given by

$$t_2 = \left[\Delta - \frac{\lambda}{2\pi}\right]\sqrt{\frac{2\pi}{\lambda g}} . \tag{2.5}$$

The total breakthrough time $t_b = t_1 + t_2$ or

$$\begin{cases} t_b = \sqrt{\dfrac{\Delta}{g}}\left\{\dfrac{1}{x} + x\left[2\log x - 1 + \log(\Delta/\xi_o)\right]\right\} , \\ x \overset{\text{def}}{=} \sqrt{\dfrac{\lambda}{2\pi\Delta}} . \end{cases} \tag{2.6}$$

Assuming all wavelengths to be present in the initial perturbation, the worst one will ultimately prevail. This is the one for which $x = x_M$, where x_M is determined by

$$(1/x_M)^2 = 1 + \log(\Delta/\xi_o) + \log(x_M)^2 , \tag{2.7}$$

and where ξ_0 is the amplitude of the initial perturbation. Clearly the worst wavelength depends on Δ/ξ_0, but not very strongly. A value of Δ/ξ_0 of order 10^3 seems reasonable. Setting $\log(\Delta/\xi_0) = 7$, Eq. (2.7) can be solved numerically for $(x_M)^2$, giving

$$\begin{cases} (x_M)^2 = 0.162 \quad , \\ \lambda_M \quad = 1.02\ \Delta\ . \end{cases} \tag{2.8}$$

This is in good agreement with the traditional lore, Eq. (1.2). Actually, the worst wavelength depends on the initial perturbation ξ_0, but when $\xi_0 \ll \Delta$ this dependence is quite weak.

What happens, of course, is that during phase (1) the shortest wavelengths grow the fastest whereas during the bubble and spike phase the worst wavelengths are the longest. These combine, as we have shown, to make $\lambda \approx \Delta$ the worst wavelength for the full composite phenomenon.

Having found x_M, we can substitute back into Eq. (2.6) to find the corresponding breakthrough time, namely

$$t_b = 4.17\ \sqrt{\Delta/g} \qquad \text{(worst mode)}\quad . \tag{2.9}$$

In this time the shell will have moved a distance s, given by

$$s = \frac{1}{2}\, g\, (t_b)^2 = 8.7\ \Delta \quad , \tag{2.10}$$

provided g is constant over this period. This is a bit less pessimistic than setting $\lambda = \Delta$ into Eq. (1.1) and writing

$$\Delta = \xi_0\, e^{\nu \bar{t}_b} \quad \text{or} \quad \bar{t}_b = \sqrt{\frac{\Delta}{2\pi g}}\ \log\left(\frac{\Delta}{\xi_0}\right) \quad . \tag{2.11}$$

Using the same value of ξ_0 as before, namely $\Delta \times 10^{-3}$, we get $\bar{t}_b = 2.8\sqrt{\Delta/g}$ and and $\bar{s} = 3.9\ \Delta$.

3. Real Fluid Effects

So far we have considered our shell to be a perfect, inviscid, and incompressible fluid without surface tension. We now shall consider, in order, the effects of surface tension, of compressibility, and of viscosity.

If the fluid possesses a surface tension T, the wavelength of maximum growth rate during the exponential phase is given by[1]

$$\lambda_M = 2\pi\sqrt{3T/(g\rho)} \quad , \tag{3.1}$$

now for normal metals T is of order 500 to 1000 in cgs units. Thus taking $T = 10^3$, $\rho = 10$, $g = 10^{12}$ we get $\lambda_M \approx 10^{-4}$ cm. This is so short compared with the worst nonlinear wavelength, namely $\lambda = \Delta$, that we can safely neglect surface tension.

Next we must consider compressibility. Normal sound speed, c_o, in metals is around 0.5×10^6 cm/s. If we simply equate this with Eq. (2.4), $v = 0.4\sqrt{g\Delta}$, and take $\Delta = 1$ mm., we find $v = c_o$ at $g = 1.56 \times 10^{13}$ cm(s)$^{-2}$. This suggests that at accelerations exceeding 10^{13}, compressibility effects might be important. In fact they are not as we shall now show. A reasonably realistic equation of state for a metal is

$$p = S\rho^{\gamma} - p_o \tag{3.2}$$

from which we get for the sound speed c_1

$$c^2 \equiv \frac{\partial p}{\partial \rho} = \frac{\gamma}{\rho}\,(p + p_o) \quad . \tag{3.3}$$

If we denote by ρ_o the density at zero pressure then

$$p_o = S\rho_o^{\gamma} \tag{3.4}$$

and at this pressure sound speed is

$$c_o^2 = \gamma p_o/\rho_o \quad . \tag{3.5}$$

For metals, γ actually varies slowly with the pressure from about 5 at low pressures to around 3.5 at a megabar or so. From Eq. (3.5) and known values for γ and c_o we find that p_o is about 1/2 to 3/4 Mbar. Now from Eq. (3.3) we have clearly

$$c > \sqrt{\gamma p/\rho} \quad . \tag{3.6}$$

But, to accelerate a shell of normal density ρ_0 and thickness Δ to an acceleration g a pressure

$$p = \rho_0 \Delta g \tag{3.7}$$

is required. Thus

$$c > (\gamma\rho_0/\rho)^{1/2} \sqrt{\Delta g} \quad . \tag{3.8}$$

Comparing this with Eq. (2.4) evaluated for $\lambda = \Delta$, we see that the bubble rise velocity is always well subsonic, provided only

$$\frac{\gamma\rho_0}{\rho} > \frac{1}{2\pi} \quad . \tag{3.9}$$

Now even for γ as small as 2, Eq. (3.9) holds for all reasonable compressions. Thus we can safely neglect compressibility.

Chandrasekhar discusses the effect of viscosity on the exponential growth phase of Rayleigh-Taylor instability. He gives[2] for the worst wavelength, λ_M, and the associated most rapid growth rate, ν_M ,

$$\begin{cases} \lambda_M = 12.80 \ (\mu^2/g\rho^2)^{1/3} \quad , \\ \nu_M = \ 0.4599 \ (\rho g^2/\mu)^{1/3} \quad , \end{cases} \tag{3.10}$$

where μ is the viscosity. If we substitute into these formulas normal values of viscosity for metals, around a centipoise at one bar, we would conclude that viscosity effects are completely negligible. However Mineev et al[3] have measured viscosities at high pressures produced by shock waves and have found μ is about 100 <u>kilo</u> poise at 1 Mbar. If we eliminate g with Eq. (3.7), the first of Eqs. (3.10) becomes

$$\lambda_M = 12.8 \left(\frac{\mu^2 \Delta}{p \ \rho}\right)^{1/3} \quad . \tag{3.11}$$

Setting in $\Delta = 0.1$ cm, $\rho = 10$, $p = 10^{12}$ dynes/cm^2, $\mu = 10^5$ poise we get $\lambda_M = 0.59$ cm. Thus with such a viscosity, the exponential phase is very different from that of the inviscid case and growth times are greatly extended.

For the same numbers as above

$$\nu_M = 2.13 \times 10^6 \tag{3.12}$$

whence, estimating breakthrough time as 7 generations [Eq. (2.11)]

$$t_b = 3.3 \times 10^{-6} \text{ s} \tag{3.13}$$

and the total distance travelled before breakup is

$$s = \frac{1}{2} g(t_b)^2 = 5.4 \text{ cm} = 54\Delta \quad , \tag{3.14}$$

a result very different from Eq. (2.10).

Finally let us consider the bubble and spike phase. The Reynold's number is defined as

$$R = av\rho/\mu \tag{3.15}$$

where a is a typical length and v a typical velocity. Setting $a = \Delta$ and the bubble rise velocity, Eq. (2.4), for v we have

$$R = \frac{\rho\Delta}{\mu} \sqrt{\frac{g\Delta}{2\pi}} = \frac{\rho\Delta}{\mu} \sqrt{\frac{p}{2\pi\rho}} \quad , \tag{3.16}$$

upon using Eq. (3.7). Using $\rho = 10$, $\Delta = 10^{-1}$, $\mu = 10^5$ and $p = 10^{12}$ gives $R = 1.26$. Now the Reynold's number is roughly the ratio of inertial to viscous forces. Thus these two forces are of about the same magnitude, so viscosity will very noticeably affect the bubble rise as well as the exponential growth before the bubble and spike phase.

Unfortunately, viscosities at shock pressures exceeding a megabar have not been measured. Up to this pressure, viscosity seems still to be rising with increased pressure, but theory would lead one to expect a turnover at some finite shock pressure. This should occur when the temperature rise from shock heating overwhelms the effect of greater shock compression. The correlation of theory with experiment is, however, at present quite unsatisfactory, so what is really needed is more measurements, especially in the 1 to 100 Mbar range. We have not made a serious attempt here to access accurately the effects of viscosity, but we have shown them to be important in the stability problem.

References

(1) S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability." Clarendon Press, Oxford 1961, p. 435.

(2) Ibid, Table XLVI, p. 447.

(3) V. N. Mineev and E. V. Savinov, "Viscosity and Melting Point of Aluminum, Lead and Sodium Chloride Subjected to Shock Compression," Sov. Phys JETP 25 No. 3, Sept. 1967, pp. 411-416.

(4) V. N. Mineev and R. M. Zaidel', "Viscosity of Water and Mercury under Shock Loading," Sov. Phys JETP 27, No. 6, Dec. 1968, pp. 874-878.

APPENDIX A

RAYLEIGH-TAYLOR INSTABILITY OF AN INVISCID FLUID PLATE.

The equations of motion of an inviscid, uncompressible fluid are

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \vec{u}.\nabla\rho = 0 \\ \nabla.\vec{u} = 0 \\ \rho\left[\dfrac{\partial\vec{u}}{\partial t} + \vec{u}.\nabla\vec{u}\right] = -\nabla p + \rho\vec{g} \end{cases} \tag{A.1}$$

where $\vec{g}$ is the direction of "gravitation" which, by the principle of equivalence mocks the acceleration. This has the static solution $\vec{u}_o = 0$ and $p = p_o$, $\rho = \rho_o$, where

$$\nabla p_o = \rho_o\vec{g}\,, \tag{A.2}$$

which we perturb by setting

$$\begin{cases} p = \rho_o + \delta p \quad , \quad p = p_o + \delta p \quad , \\ \qquad \vec{u} = \partial\vec{\xi}/\partial t \quad . \end{cases} \tag{A.3}$$

Clearly $\vec{\xi}$ represents a displacement from the static equilibrium. We suppose $\vec{\xi}$ and its derivatives to be so small that we can neglect nonlinear terms. Then the first of Eqs. (A.1) integrates to give

$$\delta\rho = -\,(\vec{\xi}.\nabla)\rho_o \tag{A.4}$$

and the second equation becomes

$$-\,\rho_o\omega^2\vec{\xi} = \nabla(\delta p) - \vec{g}(\vec{\xi}.\nabla\rho_o) \quad . \tag{A.5}$$

We have already Fourier analyzed in time, writing $\vec{\xi}(\vec{x};t) = \vec{\xi}(\vec{x})\exp[i\omega t]$. As we have no rule for calculating δp from $\vec{\xi}$ for an incompressible fluid, we eliminate it by taking the curl of Eq. (A.5), obtaining the equation of motion for $\vec{\xi}$,

$$- \omega^2 \operatorname{curl}(\rho\vec{\xi}) = - [\nabla(\vec{\xi}.\nabla\rho)] \times \vec{g} \quad . \tag{A.6}$$

As only ρ_0 enters explicitly, it causes no confusion to drop the zero subscript, as we do from here on. Now set

$$\begin{aligned} &\vec{g} = (0,0,-g) \;;\; g = \text{const.} \\ &\rho = \rho \text{ (z alone)} \\ &\vec{\xi}(x,y,z) \to \vec{\xi}(z) \exp\{i(k_x x + k_y y)\} \end{aligned} \tag{A.7}$$

and the z-component of Eq. (A.6) becomes

$$k_x\xi_y - k_y\xi_x = 0 \tag{A.8}$$

which, together with

$$\nabla.\xi \equiv (k_x\xi_x + k_y\xi_y) + \partial_z\xi_z = 0 \;, \tag{A.9}$$

gives

$$\begin{aligned} \xi_x &= (ik_x/k^2)\partial_z\xi_z \quad , \\ \xi_y &= (ik_y/k^2)\partial_z\xi_z \quad . \end{aligned} \tag{A.10}$$

Setting these values into the x- and y-components of Eq. (A.6), the two reduce to the single equation

$$\frac{d}{dz}\left[\rho\omega^2 \frac{\partial\xi}{dz}\right] - k^2\left[\rho\omega^2 + g\frac{d\rho}{dz}\right]\xi = 0 \tag{A.11}$$

where we have written ξ in place of ξ_z.

Boundary conditions are that ξ be everywhere bounded and that it be continuous. Thus across any surface that bounds two different materials we must have

$$[\![\xi]\!] = 0 \tag{A.12}$$

where ⟦....⟧ means the jump in (....) across the boundary. One other condition is needed which we get from Eq. (A.11). In each medium ρ is constant, but it jumps across boundaries. Let us replace the jump by a gradual transition zone, say extending from $z_0 - \frac{\varepsilon}{2}$ to $z_0 + \frac{\varepsilon}{2}$. Now integrate Eq. (A.11) between these limits. We get

$$\text{(A.13)} \qquad \left[\rho\omega^2\xi' - k^2\rho g\xi\right]_{z_0-\frac{\varepsilon}{2}}^{z_0+\frac{\varepsilon}{2}} + O(\varepsilon) = 0 \qquad \text{(A.13)}$$

whence, allowing ε to tend toward zero, we get

$$\text{(A.14)} \qquad [\![\ \rho\omega^2\xi' - k^2\rho g\xi\ ]\!] = 0 \qquad \text{(A.14)}$$

as our other boundary condition. Inside each medium, ρ is a constant and Eq. (A.11) therefore has the general solution

$$\text{(A.15)} \qquad \xi = Ae^{kz} + Be^{-kz} \quad . \qquad \text{(A.15)}$$

Now consider a three layered medium. For $z < -\frac{\Delta}{2}$ we have medium zero (density ρ_0); in this region

$$\xi = (A + Be^{k\Delta})e^{kz} \quad \text{for} \quad z < -\frac{\Delta}{2} \quad . \qquad \text{(A.16a)}$$

Next, for $-\frac{\Delta}{2} < z < \frac{\Delta}{2}$ we have medium 1 (density ρ_1) in which

$$\xi = Ae^{kz} + Be^{-kz} \quad \text{for} -\frac{\Delta}{2} < z < \frac{\Delta}{2} \quad . \qquad \text{(A.16b)}$$

Finally for $z > \frac{\Delta}{2}$ we have medium 2 (density ρ_2) and in this medium

$$\xi = (Ae^{k\Delta} + B)e^{-kz} \quad \text{for} \quad z > \frac{\Delta}{2} \; . \tag{A.16c}$$

We have chosen the constants so that condition (A.12) is satisfied at both interfaces and so that $\xi \to 0$ as $z \to \pm \infty$. It remains to satisfy condition (A.14) at both interfaces, $z = \pm \frac{\Delta}{2}$. These conditions may be written as

$$\begin{cases} [(\rho_2+\rho_1)\omega^2 + kg(\rho_2-\rho_1)]Ae^{k\frac{\Delta}{2}} + [(\rho_2-\rho_1)\omega^2 + kg(\rho_2-\rho_1)]Be^{-k\frac{\Delta}{2}} = 0 \\ [-(\rho_1-\rho_o)\omega^2 + kg(\rho_1-\rho_o)]Ae^{-k\frac{\Delta}{2}} + [(\rho_1+\rho_o)\omega^2 + kg(\rho_1-\rho_o)]Be^{k\frac{\Delta}{2}} = 0 \end{cases} \tag{A.17}$$

When $\frac{k\Delta}{2}$ is large enough that we may drop the terms in $\exp[-k\frac{\Delta}{2}]$ the two surfaces decouple and we have the usual dispersion relation. We are primarily interested in a relatively thin plate and in media 0 and 2, which are very tenuous, i.e., ρ_o and $\rho_2 << \rho_1$. Thus neglecting ρ_o and ρ_2, Eqs. (A.17) have a nontrivial solution only if

$$\left[\rho^2\omega^4 - k^2g^2\rho^2\right] \left[e^{k\Delta} - e^{-k\Delta}\right] = 0. \tag{A.18}$$

As $\frac{k\Delta}{2}$ is not identically zero, this yields

$$\omega^2 = \mp kg \quad ; \tag{A.19}$$

the upper sign gives the unstable modes. When these values of ω are substituted into Eqs. (A.17) we find that they reduce to

$$\begin{cases} A = 0 \ , \ \xi = Be^{-kz} \ (-\frac{\Delta}{2} < z < \frac{\Delta}{2}) \text{ for } \omega^2 = - kg, \\ \\ B = 0 \ , \ \xi = Ae^{+kz} \ (- \frac{\Delta}{2} < z < \frac{\Delta}{2}) \text{ for } \omega^2 = + kg. \end{cases} \tag{A.20}$$

Thus in the case of instability, $\omega^2 = - kg$, exactly as though the medium ρ_1 were semi-infinite. Moreover the mode structure within medium 1 (density $\rho=\rho_1$) is also completely independent of the thickness of the layer, Δ. For the stable modes, $\omega^2 = + kg$, we again have a dispersion relationship and a mode structure independent of the layer thickness Δ. The unstable modes are the Rayleigh-Taylor modes on the bottom surface, whereas the stable modes are gravity waves on the top surface.